Hw 9 (cf Q15-18 in P93,94 and Q12,14 in P111 and Lemma 8 in P105, 3rd |

1. Let f: R: —>[0,∞) be measurable. By the 2nd principle of Littlewood (one of its versions, see Q4 of Hw 7) there exists a monotonically increasing sequence φ_n of non-negative simple functions vanishing outside (-n,n) convergent a.e. to f.
   Show that, if f is also integrable then $\lim_n \int \varphi_n = \int f$ and $\lim_n \int \varphi_n(x+c)\,dx = \int f(x+c)\,dx$ for all $c \in \mathbb{R}$. Show further that $\int f(x+c)\,dx = \int f(x)\,dx,\ \forall c \in \mathbb{R}$ and

(*) $\int f(\lambda x)\,dx = \frac{1}{|\lambda|}\int f(y)\,dy \ \forall \lambda \neq 0$ (progressive, for $f = \chi_E$, $\varphi \in \mathcal{S}(-n,n)$, $0 \le f \in L(\mathbb{R})$, and general $f \in L(\mathbb{R})$. Writing $\lambda E \overset{def}{=} \{\lambda x : x \in E\}$ and $\int_b^a f \overset{def}{=} -\int_a^b f = -\int_{[a,b]} f$ if $a<b$ show that $\int_a^b f(x+c)\,dx = \int_{a+c}^{b+c} f(y)\,dy$,

and $\int_a^b f(\lambda x)\,dx = \frac{1}{\lambda}\int_{\lambda a}^{\lambda b} f(z)\,dz \quad \forall \lambda \neq 0$, e.g with $\lambda = -1$

$$LHS = \int \chi_{[a,b]}(x) f(\lambda x)\,dx = \int \chi_{-[a,b]}(-x) f(-x)\,dx = \int \chi_{[-b,-a]}(-x)\cdot f(-x)\,dx$$

$$\underset{\text{by }(*)}{=} \frac{1}{|-1|}\int \chi_{[-b,-a]}(y) f(y)\,dy = \int_{-b}^{-a} f(y)\,dy = -\int_{-a}^{-b} f(y)\,dy = RHS$$

2. A subset Z of a linear space Y with a semi-norm ($\|y\| \ge 0\ \forall y \in Y$ s.t. $\|\lambda y\| = |\lambda|\cdot\|y\|$ and $\|y_1 + y_2\| \le \|y_1\| + \|y_2\|$ $\forall \lambda \in \mathbb{R}, \forall y, y_1, y_2 \in Y$)
   Is said to be dense if for each y in Y and each positive r there exists z in Z such that $\|y - z\| < \varepsilon$.
   Show that each of the following subclasses is dense in L(R) with respect to the semi-norm

$\mathcal{S}_{00}(\mathbb{R}) := \{f :$ simple functions vanishing outside a finite interval$\}$.
$\mathcal{S}t_0(\mathbb{R}) := \{f :$ step-functions vanishing outside a finite interval$\}$.
$C_{00}(\mathbb{R}) := \{f :$ continuous functions - - — - - - - —$\}$.

(Hint: since each of the subclasses is stable with respect to lattice-operations, you need only show that each non-negative f from $\mathcal{L}(E)$ can be approximated by non-negative elements from the subclasses).

3. Try some from a subclass and make use of Q1,2 above or Littlewood's principles show the following results)Let f be an integrable function on R.)

(I) Let a_n, b_n be "Fourier coefficients" of f:

$$a_n := \int f(x)\sin nx\,dx, \qquad b_n := \int f(x)\cos nx\,dx \quad (n \in \mathbb{N}).$$

Show that $\lim_n a_n = 0 = \lim_n b_n = 0$..

(ii) $\lim_{\delta \to 0} \int |f(x) - f(x+\delta)|\,dx = 0$ (Hint: each $f \in C_{00}(\mathbb{R})$ is uniformly cts.)

4. Let f be a function of two variables (x, t) which is defined on the product Q = [a,b] x[c,d] of intervals such that for each t, the function is measurable on [a,b]. Show that:

(i) Suppose $\exists\, g \in \mathcal{L}[a,b]$ s.t. $|f(x,t)| \leq g(x)\ \forall (x,t) \in Q$. Then, $\forall t_0 \in [c,d]$,

$$\lim_{t \to t_0} \int_a^b f(x,t)\,dx = \int_a^b \left(\lim_{t \to t_0} f(x,t)\right) dx,$$

provided that, $\forall x \in [a,b]$, $\lim_{t \to t_0} f(x,t)$ exists (Hint: For $\Phi : \mathbb{R} \to \mathbb{R}$, $t_0 \in \mathbb{R}$, $\lim_{t \to t_0} \Phi(t)$ iff $\lim_n \Phi(t_n)$ exists whenever seq $t_n \to t_0$. .

(ii) $$\frac{d}{dt}\int_c^d f(x,t)\,dx = \int_c^d \frac{\partial f(x,t)}{\partial t}\,dx,$$ provided that

1) $\frac{\partial f}{\partial t}(x,t)$ exists in $\mathbb{R}$, $\forall\, x, t$

2) $\left|\frac{\partial f}{\partial t}(x,t)\right| \leq G(x)$ on Q, where $G \in \mathcal{L}[a,b]$. Hint: let $t_0 \in [c,d]$ and $t_n \to t_0$ ($t_n \neq t_0$). Let $F_n(x) := \frac{f(x,t_n) - f(x,t_0)}{t_n - t_0} \underset{Th}{\overset{\text{Mean-Value}}{=\!=}} f(x, \xi_n(x))$

where $\xi_n(x)$ lies in between t_0 and t_n. Then $|F_n(x)| \leq G(x)\ \forall x$ (and also $\lim_n F_n(x) = \frac{\partial f(x,t_0)}{\partial t}$). Hence $\int_c^d F_n(x)\,dx \to \int_c^d \frac{\partial f(x,t_0)}{\partial t}\,dx$.

5. Let F $\in$ BV[0,1] $\cap$ C[0,1] and be ABC in the interval [a,1] for each a with 0<a $\leq$1. Show that f is ABC on [0,1].
(Hint: Use the continuity of the indefinite integral defined by F', and also use the fundamental theorem of calculus applied to
And finally pass to the limit as (F is continuous at 0).

5. Show that ABC[a,b] is stable w.r.t linear operations and multiplication (also quotient f/g is g is bounded away from zero by a positive constant). Show the validity of "integration by parts":

6 (Two runners' Lemma). Let f, g Be integrable on [a,b] such that .. for each x in [a,b]. Show that f = g a.e. (If two runners always side-by-side (same distances from the starting a) then their speeds are the same a.e.)
(Hint. Let h=f-g. Then the integrals of h over any open interval, any open set, any closed sets contained in [a,b] are zero. Let P:= { x: h(x) > 0} and let B be a closed set contained in P. Then h is strictly positive on B and the integral of h over B is zero so B must be of measure zero. By Littlewood P is also of measure zero.)